JOURNAL OF APPROXIMATION THEORY 44, 277–288 (1985)

Tchebycheff Approximation of Continuous Functions by Harmonic Polynomials on Conic Sections

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Communicated by Oved Shisha

Received May 18, 1984

The problem of finding a best Tchebycheff approximation to a given continuous function f, defined on a compact portion of a plane conic section, from the set of harmonic polynomials of degree n or less is studied. It is shown that the Haar condition is fulfilled by such harmonic polynomials. Interesting relationships which exist between this problem and certain classical approximation problems are explored. Numerical examples are given to illustrate the theory.

1. INTRODUCTION

The Tchebycheff or uniform approximation of a continuous function f defined on a compact domain X is a classical problem in approximation theory. In practice the set S of approximating functions is an N-dimensional linear subspace of the space $C(X)$ of continuous functions on X with uniform norm

$$\|f\| = \max\{|f(x)| : x \in X\}. \tag{1.1}$$

In order for the approximation problem to be tractable, it is required that S satisfy the Haar condition, that is, that the only function in S which has N or more zeros on X is the zero function. In this case, each $f \in C(X)$ has a unique best approximation $s^* \in S$ [2, p. 80], and the error function $E = f - s^*$ is characterized by the alternation property [2, p. 75].

The aim of this paper is to show that the space H_n of harmonic polynomials of degree n or less satisfies the Haar condition when the domain X is restricted to be a compact subset of a conic section in $\mathbb{R}^2$. A mild regularity condition must be imposed in the hyperbolic case. This allows some classical examples of Haar subspaces to be unified under one framework and provides many interesting numerical examples.

0021-9045/85 $3.00

2. The Haar Subspace of Harmonic Polynomials

A polynomial $h(x, y)$ is said to be harmonic if it satisfies Laplace's equation

$$\Delta h = h_{xx} + h_{yy} = 0. \tag{2.1}$$

The functions

$$\begin{aligned} h_{2\nu-1}(x, y) &= \operatorname{Im} z^\nu, \qquad \nu = 1, 2, \ldots, n, \\ h_{2\nu}(x, y) &= \operatorname{Re} z^\nu, \qquad \nu = 0, 1, \ldots, n, \end{aligned} \tag{2.2}$$

where $z = x + iy$, form a basis for the $N = (2n+1)$-dimensional space H_n of all harmonic polynomials of degree n or less. Note that these basis functions can be generated by the recursion formulas

$$h_{2\nu-1}(x, y) = yh_{2\nu-2}(x, y) + xh_{2\nu-3}(x, y), \tag{2.3}$$

$$h_{2\nu}(x, y) = xh_{2\nu-2}(x, y)) - yh_{2\nu-3}(x, y), \tag{2.4}$$

$\nu = 1, 2, \ldots, n$, where $h_{-1}(x, y) = 0$, $h_0(x, y) = 1$.

To guarantee that H_n satisfies the Haar condition, the domain of definition for $h \in H_n$ must be restricted. Bezout's theorem from algebraic geometry shows how this can be accomplished [6, p. 59].

THEOREM 1 (Bezout's theorem). *Let $p(x, y)$, $q(x, y)$ be real valued polynomials of the variables x, y with degrees m, n, respectively. If p and q have no nonconstant common factors, then the system of equations*

$$\begin{aligned} p(x, y) &= 0, \\ q(x, y) &= 0, \end{aligned} \tag{2.5}$$

can have at most mn solutions $(x, y) \in \mathbb{R}^2$.

When polynomials p, q have no nonconstant common factors they are said to be relatively prime. We shall need the following special case of the theorem where p has degree 2 and where q has degree n or less.

COROLLARY. *Let*

$$p(x, y) = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F, \tag{2.6}$$

where $A, B, \ldots, F \in \mathbb{R}$. The space H_n of all harmonic polynomials of degree n or less satisfies the Haar condition on $p(x, y) = 0$ if and only if each harmonic polynomial $h \in H_n$ is relatively prime to p.

To use the corollary of Theorem 1, we must be able to classify those quadratic polynomials p which are relatively prime to each $h \in H_n$. To rule out trivial cases, a covering hypothesis will be made about the coefficients of (2.6). Let

$$\Delta_1 = A + C, \tag{2.7}$$

$$\Delta_2 = AC - B^2, \tag{2.8}$$

$$\Delta_3 = \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix}. \tag{2.9}$$

We shall henceforth assume that $\Delta_3 \neq 0$ and either

(i) $\Delta_2 > 0$ with $\Delta_1 \Delta_3 < 0$, or

(ii) $\Delta_2 = 0$, or

(iii) $\Delta_2 < 0$,

so that the equation $p(x, y) = 0$ represents a real ellipse, parabola, or hyperbola, respectively [5 p. 70].

A preparatory lemma is needed to simplify the presentation of the classification theorem.

LEMMA. *Let $n \geqslant 2$ and let $p(x, y)$ be given by* (2.6) *with*

$$B = 0, \qquad |A| + |C| > 0. \tag{2.10}$$

For each $k = 0, 1, \ldots, n-2$ consider the two nonhomogeneous discrete boundary value problems

$$\begin{aligned} &A\alpha_{2j} + C\alpha_{2j-2} = (-1)^j \binom{k+2}{2j}, \qquad j = 0, 1, \ldots, l \\ &\alpha_{-2} = \alpha_{2l} = 0, \qquad \text{where } l = [\![(k+2)/2]\!] \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} &A\alpha_{2j+1} + C\alpha_{2j-1} = (-1)^j \binom{k+2}{2j+1}, \qquad j = 0, 1, \ldots, m \\ &\alpha_{-1} = \alpha_{2m+1} = 0, \qquad \text{where } m = [\![(k+1)/2]\!]. \end{aligned} \tag{2.12}$$

(Here $[\![\cdot]\!]$ is the greatest integer function and $(\cdot)$ is the binomial coefficient.) If neither (2.11) nor (2.12) has a solution for any $k = 0, 1, \ldots, n-2$, then p is relatively prime to each $h \in H_n$.

Proof. By way of contradiction, suppose that p is not relatively prime to some $h \in H_n$. We will show that either (2.11) or (2.12) must have a solution for some $k = 0, 1, 2, \ldots, n-2$.

In view of the blanket hypothesis, p has no linear factors, so there exists a polynomial q of degree k, $0 \leqslant k \leqslant n-2$, such that

$$h(x, y) = p(x, y)\, q(x, y).$$

Decompose p and q into the forms

$$\begin{aligned} p(x, y) &= p_2(x, y) + p_1(x, y) + p_0(x, y), \\ q(x, y) &= q_k(x, y) + q_{k-1}(x, y) + \cdots + q_0(x, y), \end{aligned}$$

where p_i is homogeneous of degree i, $i = 0, 1, 2$ and q_j is homogeneous of degree j, $j = 0, 1, \ldots, k$. Let

$$q_k(x, y) = \beta_0 x^k + \beta_1 x^{k-1} y + \cdots + \beta_k y^k. \tag{2.13}$$

The product $p_2 q_k$ is homogeneous of degree $k+2$ and harmonic, so there exist real numbers a, b, not both zero, such that

$$\begin{aligned} p_2(x, y)\, q_k(x, y) &= a \cdot h_{2k+3}(x, y) + b \cdot h_{2k+4}(x, y) \\ &= a \cdot \left[\sum_{j=0}^{m} (-1)^j \binom{k+2}{2j+1} x^{k-2j+1} y^{2j+1} \right] \\ &\quad + b \cdot \left[\sum_{j=0}^{l} (-1)^j \binom{k+2}{2j} x^{k-2j+2} y^{2j} \right], \end{aligned} \tag{2.14}$$

where $m = [\![(k+1)/2]\!]$ and $l = [\![(k+2)/2]\!]$. On the other hand, by using (2.13), we can also write

$$\begin{aligned} p_2(x, y)\, q_k(x, y) &= [Ax^2 + Cy^2] \left[\sum_{j=0}^{k} \beta_j x^{k-j} y^j \right] \\ &= \sum_{j=0}^{k+2} [A\beta_j + C\beta_{j-2}]\, x^{k-j+2} y^j, \end{aligned} \tag{2.15}$$

where $\beta_{-2} = \beta_{-1} = \beta_{k+1} = \beta_{k+2} = 0$. Upon equating coefficients in (2.14) and (2.15) we find that $\beta_{-2}, \beta_0, \ldots, \beta_{2l}$ must satisfy the difference equation

$$A\beta_{2j} + C\beta_{2j-2} = (-1)^j \binom{k+2}{2j} b, \qquad j = 0, 1, \ldots, l. \tag{2.16}$$

Similarly, $\beta_{-1}, \beta_1, \ldots, \beta_{2m+1}$ must satisfy the difference equation

$$A\beta_{2j+1} + C\beta_{2j-1} = (-1)^j \binom{k+2}{2j+1} a, \qquad j = 0, 1, \ldots, m. \tag{2.17}$$

Since a, b are not both zero we may scale either (2.16) by $1/b$ or (2.17) by $1/a$ to obtain a solution for either (2.12) or (2.11).

THEOREM 2. *Let*

$$p(x, y) = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0 \tag{2.18}$$

define an ellipse, a parabola, or a hyperbola in $\mathbb{R}^2$. *When* (2.18) *defines a hyperbola, assume further that*

$$[(A + C) + 2i(B^2 - AC)^{1/2}]^{k+2} \notin \mathbb{R}, \qquad k = 0, 1, \ldots, n-2. \tag{2.19}$$

Then the space H_n *satisfies the Haar condition on the curve* (2.18), *i.e., no* h, $h \neq 0$, *from the* $N = (2n+1)$*-dimensional space* H_n *can have more than* $2n$ *zeros on* (2.18).

Proof. By the covering assumptions, p is relatively prime to each $h \in H_1$. We must show that p is relatively prime to any given $h \in H_{k+2}$, $0 \leqslant k \leqslant n-2$. In so doing we shall first work under the assumption that $B = 0$ and then remove this restriction.

Assume then that $B = 0$. We shall use the previous lemma and show that neither (2.11) nor (2.12) can have a solution. Clearly, if $A = 0$, then neither (2.11) nor (2.12) can have a solution, so we may assume $A > 0$. To facilitate the presentation we shall also assume that k is even. When k is odd the same analysis given when k is even can be carried through with only minor changes in detail. Finally, we assume that one of (2.11) or (2.12) has a solution and derive a contradiction to the hypothesis of the theorem.

If (2.11) has a solution, then after solving successively for $\alpha_0, \alpha_2, \ldots$, we find

$$\alpha_{k+2} = (-1)^l \left[\sum_{j=0}^{l} \binom{k+2}{2j} (C/A)^{(k-2j+2)/2} \right] \Big/ A = 0, \tag{2.20}$$

where $l = (k+2)/2$. Analogously if (2.12) has a solution, then after solving successively for $\alpha_1, \alpha_3, \ldots$, we find

$$\alpha_{k+1} = (-1)^m \left[\sum_{j=0}^{m} \binom{k+2}{2j+1} (C/A)^{(k+2j)/2} \right] \Big/ A = 0, \tag{2.21}$$

where $m=k/2$. If $C \geqslant 0$, neither (2.20) nor (2.21) can hold. If $C<0$, then

$$\alpha_{k+2}=(-1)^l \operatorname{Re}[1+i(-C/A)^{1/2}]^{k+2}/A=0,$$

and

$$\alpha_{k+1}=(-1)^m (-C/A)^{1/2} \operatorname{Im}[1+i(-C/A)^{1/2}]^{k+2}/A=0.$$

Since

$$\begin{aligned} 2 \operatorname{Re}[1+i(-C/A)^{1/2}]^{k+2} \operatorname{Im}[1+i(-C/A)^{1/2}]^{k+2} \\ = \operatorname{Im}[(A+C)+2i(-AC)^{1/2}]^{k+2}/A^{k+2}, \end{aligned}$$

we have that one of (2.20) or (2.21) holds only if

$$[(A+C)+2i(-AC)^{1/2}]^{k+2} \in \mathbb{R}. \tag{2.22}$$

This shows that for the special case when $B=0$, the space H_n satisfies the Haar condition on (2.18) provided that (2.19) holds when (2.18) defines a hyperbola.

As a final step we must remove the restriction on B. This is accomplished by applying the orthogonal rotation

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & -\beta & 0 \\ \beta & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}, \qquad \alpha^2+\beta^2=1 \tag{2.23}$$

to the equation

$$h(x, y)=p(x, y)\, q(x, y)$$

to obtain

$$h'(x', y')=p'(x', y')\, q'(x', y'),$$

where $p'(x', y')=ax'^2+cy'^2+2dx'+2ey'+f$. Under the transformation (2.23), $a+c=A+C$ and $ac=AC-B^2$. Moreover, since (2.23) preserves harmonic polynomials, the special case of the theorem which has been established for the polynomials h', p', and q' can be applied to obtained the desired generalization.

We now present a simple example which illustrates the approximation of a continuous function by a harmonic polynomial on a conic section.

EXAMPLE 1. Suppose we wish to find the best uniform approximation

$h(x, y) = a_0 + a_1 y + a_2 x$, $a_0, a_1, a_2 \in \mathbb{R}$, from H_1 to the funcion $f(x, y) = y^2$ on the ellipse

$$X = \{(x, y) \in \mathbb{R}: x^2 + y^2/4 = 1\}.$$

Restricting f to the ellipse X produces a closed curve $\Gamma = \{(x, y, z): z = y^2, (x, y) \in X\}$ in $\mathbb{R}^3$. The approximation problem is viewed as finding the plane $z = a_0 + a_1 y + a_2 x$ which is "closest" to Γ in the Tchebycheff sense. The plane $z = 2$ yields the best Tchebycheff approximation to Γ. This is verified by noting that the error curve

$$E(x, y) = f(x, y) - 2h_0(x, y) = y^2 - 2$$

has four extremal points on X at $(\pm 1, 0)$, $(0, \pm 2)$ and thus alternates three times on X.

We shall now point out several interesting relationships which exist between certain classical Haar spaces and the space of harmonic polynomials restricted to a conic section. In so doing we shall see how the theory of harmonic polynomial approximation can serve to unify and generalize many of the classical approximation problems.

EXAMPLE 2. Let the space H_n of all harmonic polynomials of degree n or less be restricted to the unit circle

$$X = \{(x, y) \in \mathbb{R}: x^2 + y^2 = 1\}.$$

Using the polar transformation $x = \cos(t)$, $y = \sin(t)$, $0 \leqslant t < 2\pi$, we see that

$$h_{2\nu}(x, y) = \cos(\nu t), \qquad \nu = 0, 1, ..., n,$$
$$h_{2\nu-1}(x, y) = \sin(\nu t), \qquad \nu = 1, 2, ..., n,$$

i.e., harmonic polynomials can be used to generate the space of trigonometric polynomials of degree n or less.

More generally, if H_n is restricted to a connected segment of a conic section parametrized in the form

$$r(t) = \frac{\rho}{1 - \varepsilon \cos(t)}, \qquad \rho > 0, \varepsilon \geqslant 0, \tag{2.24}$$

then the basis functions for H_n become the weighted trigonometric functions

$$h_{2\nu}(x, y) = r^{\nu}(t) \cos(\nu t), \qquad \nu = 0, 1, ..., n,$$
$$h_{2\nu-1}(x, y) = r^{\nu}(t) \sin(\nu t), \qquad \nu = 1, ..., n.$$

Condition (2.19), which must be imposed when (2.24) defines a hyperbola, that is when $\varepsilon > 1$, can be expressed as

$$[(2-\varepsilon^2)+2i(\varepsilon^2-1)^{1/2}]^{k+2} \notin \mathbb{R}, \qquad k=0, 1,..., n-2. \tag{2.25}$$

EXAMPLE 3. The space P_n of all polynomials in one variable of degree n or less can be generated from the space H_n of all harmonic polynomials of degree n or less, albeit somewhat unnaturally due to the fact that the dimension of H_n is odd, $N=2n+1$, while no restriction needs to be placed on $N=n+1$, the dimension of P_n.

One way to generate P_n from H_n is to restrict H_n to the parabola

$$X=\{(x, y) \in \mathbb{R}: y=x^2\}. \tag{2.26}$$

On X the recursion formulas (2.3), (2.4) can be used to show that if $h_{2\nu-3}$, $h_{2\nu-2}$ are polynomials in x of degree ν, $\nu-1$, respectively, then $h_{2\nu-1}$, $h_{2\nu}$ are polynomials in x of degree $\nu+1$, $\nu+2$, respectively. Therefore, if the basis functions for H_n are ordered according to the pattern h_0, h_2, h_1, h_3, h_4, h_6, h_5, h_7,..., then at any stage the first $n+1$ of these functions form a basis for P_n.

The spaces H_n and P_n can also be related by restricting H_n to a straight line in $\mathbb{R}^2$. Indeed let

$$\begin{aligned} x &= x(t) = \gamma_1 t + \gamma_2, \\ y &= y(t) = \gamma_3 t + \gamma_4, \end{aligned} \tag{2.27}$$

$|\gamma_1|+|\gamma_3|>0$, be a parametric representation of a straight line. Substituting (2.27) into the basis functions for H_n produces polynomials in t. Under this restriction H_n suffers a loss of dimension. In the next section an example is presented which shows the behavior of best approximations from $H_n(X)$ when an elliptical domain X collapses to form a line segment. Quite interesting phenomena occur.

EXAMPLE 4. Harmonic polynomials can also be used to generate a class of rational functions which satisfy the Haar condition. Indeed, consider a hyperbola of the form

$$Ax^2+2Bxy+2Dx+2Ey+F=0 \tag{2.28}$$

with $[A+2iB]^j \notin \mathbb{R}$ for $j=2,..., n$ so that the hypothesis of Theorem 2 is fulfilled for H_n. We solve (2.28) for y to obtain

$$y(x)=P(x)/(2Q(x)), \tag{2.29}$$

where $P(x) = -[Ax^2 + 2Dx + F]$, $Q(x) = [Bx + E]$. By using (2.29) for y in the $2n+1$ basis functions (2.2) for H_n we obtain the system of rational functions

$$\begin{aligned} h_0(x, y) &= 1, \\ h_{2\nu-1}(x, y) &= R_{2\nu-1}(x)/S_\nu(x), \qquad \nu = 1,..., n, \\ h_{2\nu}(x, y) &= R_{2\nu}(x)/S_\nu(x), \qquad \nu = 1,..., n, \end{aligned}$$

where $S_\nu(x) = Q^\nu(x)$, $\nu = 1,..., n$ and where $R_{2\nu-1}(x)$, $R_{2\nu}(x)$ are polynomials in x of degree 2ν or less, $\nu = 1,..., n$.

3. Numerical Examples

In this section we shall present numerical examples of the approximation problem outlined in Section 2. A Remez type algorithm was used to carry out the computation [1].

Example 1. We approximate the function

$$f(x, y) = \exp(x + y) \tag{3.1}$$

by a harmonic polynomial

$$\begin{aligned} h(x, y) &= a_0 h_0(x, y) + \cdots + a_4 h_4(x, y) \\ &= a_0 + a_1 y + a_2 x + a_3(2xy) + a_4(x^2 - y^2) \end{aligned} \tag{3.2}$$

from H_2 on the sequence of ellipses

$$x^2 + \mu y^2 - 1 = 0, \qquad \mu = 1, \quad 10, \quad 10^2, \quad 10^4, \quad 10^8 \tag{3.3}$$

which collapse to the interval $-1 \leqslant x \leqslant 1$ as $\mu \to \infty$. The errors $\| f - h^* \|$, optimal parameters, and extremal points are shown in Table I. The best approximations display appropriate behavior in that they converge to the best approximation to $f(x, 0) = \exp(x)$ by a polynomial p of degree 2 or less on $[-1, 1]$ as the ellipses collapse to $[-1, 1]$. Indeed, we have used the basis 1, x, $2x^2 - 1$ of Tchebycheff polynomials on $[-1, 1]$ to compute the best approximation to $f(x, 0) = \exp(x)$ from the space of polynomials of degree 2 or less on $[-1, 1]$, and found the best approximation to be given by

$$\begin{aligned} p^*(x) &= 1.2660 + 1.1302x + 0.2770\ (2x^2 - 1) \\ &= 0.9890 + 1.1302x + 0.5540x^2 \end{aligned} \tag{3.4}$$

TABLE I

Tchebycheff Approximation of $f(x, y) = \exp(x+y)$ on the Sequence of Ellipses $x^2 + \mu y^2 - 1 = 0$, $\mu = 1, 10, 10^2, 10^4, 10^8$ by a Harmonic Polynomial $h \in H_2$

μ	$\|f-h^*\|$	Parameters	(x, y)-in extremal set
1	0.137363	$a_0 = 1.5560$	(0.7071, 0.7071)
		$a_1 = 1.2712$	(−0.1622, 0.9867)
		$a_2 = 1.2712$	(−0.9354, 0.3536)
		$a_3 = 0.6121$	(−0.7071, −0.7071)
		$a_4 = 0.0000$	(0.3536, −0.9354)
			(0.9867, −0.1622)
10	0.052335	$a_0 = 1.0884$	(0.9535, 0.0953)
		$a_1 = 1.1438$	(0.2872, 0.3029)
		$a_2 = 1.1438$	(−0.6852, 0.2303)
		$a_3 = 0.5597$	(−0.9535, −0.0953)
		$a_4 = 0.4579$	(−0.1420, −0.3130)
			(0.7859, −0.1955)
10^2	0.045259	$a_0 = 1.0002$	(1.0000, 0.0000)
		$a_1 = 1.1308$	(0.4709, 0.0882)
		$a_2 = 1.1308$	(−0.5254, 0.0851)
		$a_3 = 0.5529$	(−0.9950, −0.0099)
		$a_4 = 0.5420$	(0.3464, −0.0938)
			(0.6363, −0.0771)
10^4	0.045020	$a_0 = 0.9892$	(1.0000, 0.0000)
		$a_1 = 1.1302$	(0.5516, 0.0083)
		$a_2 = 1.1302$	(−0.4458, 0.0089)
		$a_3 = 0.5540$	(−0.9999, −0.0001)
		$a_4 = 0.5539$	(−0.4278, −0.0090)
			(0.5683, −0.0082)
10^8	0.045017	$a_0 = 0.9890$	(1.0000, 0.0000)
		$a_1 = 1.1302$	(0.5600, 0.0001)
		$a_2 = 1.1302$	(−0.4371, 0.0001)
		$a_3 = 0.5540$	(−1.0000, 0.0000)
		$a_4 = 0.5540$	(−0.4369, −0.0001)
			(0.5600, −0.0001)

with $\|f-p^*\| = 0.045017$. The extremal points of the error curve are located at -1.000, -0.437, 0.560, and 1.000. On the other hand, we see from Table I that for $\mu = 10^8$ the best approximation $h^* \in H_2$ can be written as

$$h^*(x, y) = 0.9890 + 1.1302(x+y) + 0.5540(x^2 + 2xy - y^2) \tag{3.5}$$

with $\|f-h^*\| = 0.045017$. Furthermore, on each ellipse there are six extremal points. As $\mu \to \infty$ we see that one extremal point converges to $(-1, 0)$ while another converges to $(1, 0)$. Also, one pair of extremal points converges from above and below to $(-0.437, 0.0000)$ while the other pair converges from above and below to $(0.560, 0.000)$.

EXAMPLE 2. We again consider the problem of approximating the function

$$f(x, y) = \exp(x + y) \tag{3.6}$$

by a harmonic polynomial $h \in H_2$, but now we restrict the domain X to be the "top half" of one of the ellipses

$$x^2 + \mu y^2 - 1 = 0, \qquad \mu = 1, \quad 10, \quad 10^2, \quad 10^4, \quad 10^8, \quad y \geqslant 0 \tag{3.7}$$

TABLE II

Tchebycheff Approximation of the Function $f(x, y) = \exp(x + y)$ by a Harmonic Polynomial $h \in H_2$ on the "Top Half" of the Ellipses $x^2 + \mu y^2 - 1 = 0$, $y > 0$, $\mu = 1, 10, 10^2, 10^4, 10^8$

μ	$\|f - h^*\|$	Parameters	(x, y)-in extremal set
1	0.041130	$a_0 = 1.2494$	(1.0000, 0.0000)
		$a_1 = 1.7999$	(0.9559, 0.2936)
		$a_2 = 1.1341$	(0.5960, 0.8030)
		$a_3 = 0.7568$	(−0.1074, 0.9942)
		$a_4 = 0.2937$	(−0.7870, 0.6169)
			(−1.0000, 0.0000)
10	0.014981	$a_0 = 0.9009$	(1.0000, 0.0000)
		$a_1 = 1.7069$	(0.8807, 0.1498)
		$a_2 = 1.1902$	(0.3182, 0.2998)
		$a_3 = 0.3753$	(−0.4640, 0.2801)
		$a_4 = 0.6422$	(−0.9397, 0.1082)
			(−1.0000, 0.0000)
10^2	0.014420	$a_0 = 0.9547$	(1.0000, 0.0000)
		$a_1 = 1.5377$	(0.9286, 0.0371)
		$a_2 = 1.1896$	(0.4422, 0.0897)
		$a_3 = -0.1561$	(−0.3634, 0.0932)
		$a_4 = 0.5884$	(−0.9197, 0.0393)
			(−1.0000, 0.0000)
10^4	0.013602	$a_0 = 1.0077$	(1.0000, 0.0000)
		$a_1 = -0.2621$	(0.9398, 0.0034)
		$a_2 = 1.1888$	(0.4870, 0.0087)
		$a_3 = -6.5802$	(−0.3162, 0.0095)
		$a_4 = 0.5354$	(−0.9074, 0.0042)
			(−1.0000, 0.0000)
10^8	0.013491	$a_0 = 1.0145$	(1.0000, 0.0000)
		$a_1 = -197.07$	(0.9409, 0.0000)
		$a_2 = 1.1887$	(0.4919, 0.0001)
		$a_3 = -711.34$	(−0.3107, 0.0001)
		$a_4 = 0.5285$	(−0.9059, 0.0000)
			(−1.0000, 0.0000)

with endpoints $(-1, 0)$, $(1, 0)$. The errors $\| f - h^* \|$, optimal parameters, and extremal points are shown in Table II.

An interesting anomaly occurs here. Indeed, considerations of continuity might at first lead us to expect that the best approximations h^* would converge as $\mu \to \infty$ to the best approximation p^* as in previous example, but they do not. After a moment's reflection it is not difficult to see why the best approximation h^* cannot converge to the best approximation p^*. Indeed, let

$$E(x, y, \mu) = f(x, y) - h^*(x, y), \qquad 1 \leqslant \mu < \infty,$$

denote the error in the best approximation to f on the elliptic arc $X_\mu = \{(x, y) \in \mathbb{R}^2 \colon x^2 + \mu y^2 - 1 = 0,\ y \geqslant 0\}$ and let

$$E(x, 0, \infty) = f(x, 0) - p^*(x)$$

denote the error in the best approximation to $f(x, 0)$ from the space of polynomials of degree 2 or less on the interval $[-1, 1]$. For $1 \leqslant \mu < \infty$, the error curve $E(x, y, \mu)$ has six extremal points and alternates five times on X_μ, while the error curve $E(x, 0, \infty)$ has four extremal points and alternates three times on $[-1, 1]$. Since $y \geqslant 0$ on X_μ no pair of extremal points of $E(x, y, \mu)$ can converge to a single extremal point of $E(x, 0, \infty)$ as occurred in the previous example. Thus $E(x, y, \mu)$ cannot converge to $E(x, 0, \infty)$ as $\mu \to \infty$. As μ increases from $\mu = 1$ to $\mu = 10^8$ the error decreases monotonically to 0.013491. This limiting error is significantly less than the limiting error of 0.045017 of Example 1.

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